1. Faults occur in a roll of material at a rate of λ per m^2. To estimate λ, three pieces of material of sizes 3 m^2, 7 m^2 and 10 m^2 are selected and the number of faults X_1, X_2 and X_3 respectively are recorded.

The estimator $\hat{\lambda}$, where

$$\hat{\lambda} = k(X_1 + X_2 + X_3)$$

is an unbiased estimator of λ.

(a) Write down the distributions of X_1, X_2 and X_3 and find the value of k.

(4)

(b) Find Var($\hat{\lambda}$).

(3)

A random sample of n pieces of this material, each of size 4 m^2, was taken. The number of faults on each piece, Y, was recorded.

(c) Show that $\frac{1}{4}\overline{Y}$ is an unbiased estimator of λ.

(2)

(d) Find Var($\frac{1}{4}\overline{Y}$).

(3)

(e) Find the minimum value of n for which $\frac{1}{4}\overline{Y}$ becomes a better estimator of λ than $\hat{\lambda}$.

(2)
(Total 14 marks)

2. The weights of the contents of jars of jam are normally distributed with a stated mean of 100 g. A random sample of 7 jars was taken and the contents of each jar, x grams, was weighed. The results are summarised by the following statistics.

$$\sum x = 710.9, \quad \sum x^2 = 72\ 219.45.$$

Test at the 5% level of significance whether or not there is evidence that the mean weight of the contents of the jars is greater than 100 g. State your hypotheses clearly.

(Total 8 marks)

3. An engineer decided to investigate whether or not the strength of rope was affected by water. A random sample of 9 pieces of rope was taken and each piece was cut in half. One half of each piece was soaked in water overnight, and then each piece of rope was tested to find its strength. The results, in coded units, are given in the table below

Rope no.	1	2	3	4	5	6	7	8	9
Dry rope	9.7	8.5	6.3	8.3	7.2	5.4	6.8	8.1	5.9
Wet rope	9.1	9.5	8.2	9.7	8.5	4.9	8.4	8.7	7.7

Assuming that the strength of rope follows a normal distribution, test whether or not there is any difference between the mean strengths of dry and wet rope. State your hypotheses clearly and use a 1% level of significance.

(Total 8 marks)

4. A certain vaccine is known to be only 70% effective against a particular virus; thus 30% of those vaccinated will actually catch the virus. In order to test whether or not a new and more expensive vaccine provides better protection against the same virus, a random sample of 30 people were chosen and given the new vaccine. If fewer than 6 people contracted the virus the new vaccine would be considered more effective than the current one.

(a) Write down suitable hypotheses for this test.

(1)

(b) Find the probability of making a Type I error.

(2)

(c) Find the power of this test if the new vaccine is

(i) 80% effective,

(ii) 90% effective.

(3)

An independent research organisation decided to test the new vaccine on a random sample of 50 people to see if it could be considered more than 70% effective. They required the probability of a Type I error to be as close as possible to 0.05.

(d) Find the critical region for this test.

(2)

(e) State the size of this critical region.

(1)

(f) Find the power of this test if the new vaccine is

(i) 80% effective,

(ii) 90% effective.

(2)

(*g*) Give one advantage and one disadvantage of the second test.

(2)

(Total 13 marks)

5. Gill, a member of the accounts department in a large company, is studying the expenses claims of company employees. She assumes that the claims, in £, follow a normal distribution with mean μ and variance σ^2. As a first stage in her investigation she took the following random sample of 10 claims.

30.85, 99.75, 142.73, 223.16, 75.43, 28.57, 53.90, 81.43, 68.62, 43.45.

(a) Find a 95% confidence interval for μ.

(6)

The chief accountant would like a 95% confidence interval where the difference between the upper confidence limit and the lower confidence limit is less than 20.

(b) Show that $\frac{\sigma^2}{n} < 26.03$ (to 2 decimal places), where n *is* the size of the sample required to achieve this.

(3)

Gill decides to use her original sample of 10 to obtain a value for σ^2 so that the chance of her value being an underestimate is 0.01.

(c) Find such a value for σ^2.

(3)

(d) Use this value for σ^2 to estimate the size of sample the chief accountant requires.

(2)
(Total 14 marks)

6. An educational researcher is testing the effectiveness of a new method of teaching a topic in mathematics. A random sample of 10 children were taught by the new method and a second random sample of 9 children, of similar age and ability, were taught by the conventional method. At the end of the teaching, the same test was given to both groups of children.

The marks obtained by the two groups are summarised in the table below.

	New method	Conventional method
Mean ($\bar{x}$)	82.3	78.2
Standard deviation (s)	3.5	5.7
Number of students (n)	10	9

(a) Stating your hypotheses clearly and using a 5% level of significance, investigate whether or not

(i) the variance of the marks of children taught by the conventional method is greater than that of children taught by the new method,

(4)

(ii) the mean score of children taught by the conventional method is lower than the mean score of those taught by the new method.

(6)

[In each case you should give full details of the calculation of the test statistics.]

(b) State any assumptions you made in order to carry out these tests.

(1)

(c) Find a 95% confidence interval for the common variance of the marks of the two groups.

(5)

(Total 16 marks)

7. A statistics student is trying to estimate the probability, p, of rolling a 6 with a particular die. The die is rolled 10 times and the random variable X_1 represents the number of sixes obtained.

The random variable $R_1 = \dfrac{X_1}{10}$ is proposed as an estimator of p.

(a) Show that R_1 is an unbiased estimator of p.

(1)

The student decided to roll the die again n times ($n > 10$) and the random variable X_2 represents the number of sixes in these n rolls. The random variable $R_2 = \dfrac{X_2}{n}$ and the random variable $Y = \frac{1}{2}(R_1 + R_2)$.

(b) Show that both R_2 and Y are unbiased estimators of p.

(2)

(c) Find Var (R_2) and Var (Y).

(3)

(d) State giving a reason which of the 3 estimators R_1, R_2 and Y are consistent estimators of p.

(2)

(e) For the case $n = 20$ state, giving a reason, which of the 3 estimators R_1, R_2 and Y you would recommend.

(4)

The student's teacher pointed out that a better estimator could be found based on the random variable $X_1 + X_2$.

(f) Find a suitable estimator and explain why it is better than R_1, R_2 and Y.

(6)
(Total 18 marks)

1. (a) $X_1 \sim \text{Po}(3\lambda)$

$X_2 \sim \text{Po}(7\lambda)$ M1

$X_3 \sim \text{Po}(10\lambda)$

$\text{E}(\hat{\lambda}) = k\,[\text{E}(X_1) + \text{E}(X_2) + \text{E}(X_3)]$ M1

$= 20\lambda k$

$\hat{\lambda}$ unbiased therefore $20\lambda k = \lambda$ M1

$k = \dfrac{1}{20}$ A1 4

Note

M1 all 3 needed. Poisson and mean
M1 adding their means
M1 putting their $\text{E}(\hat{\lambda}) = \lambda$
A1 cao

(b) $\text{Var}(\hat{\lambda}) = \dfrac{1}{20^2}\text{Var}(X_1 + X_2 + X_3)$ M1

$= \dfrac{1}{20^2}(3\lambda + 7\lambda + 10\lambda)$ M1

$= \dfrac{\lambda}{20}$ A1ft 3

Note

M1 use of $k^2\text{Var}(X_1 + X_2 + X_3)$
M1 using their means from part(a) as Variances and adding together
A1 cao

(c) $Y \sim \text{Po}(4\lambda)$

$\text{E}\left(\frac{1}{4}\bar{Y}\right) = \frac{1}{4} \times 4\lambda = \lambda$ therefore unbiased — M1 A1 — 2

Note

M1 use of 4λ
A1 cso plus conclusion. Accept working out bias to = 0

(d) $\text{Var}\left(\frac{1}{4}\bar{Y}\right) = \frac{1}{16} \times \frac{4\lambda}{n}$ — M1 B1

$= \frac{\lambda}{4n}$ — A1 — 3

Note

M1 $\frac{1}{16} \times \text{Var}\,\bar{Y}$

B1 for $\text{Var}\,\bar{Y} = \frac{4\lambda}{n}$

A1 cao

(e) $\frac{\lambda}{4n} < \frac{\lambda}{20}$ — M1

$n > 5$ therefore $n = 6$ — A1 — 2

Note

M1 for $\text{Var}\left(\frac{1}{4}\bar{Y}\right) > \text{Var}(\hat{\lambda})$

A1 n = 6

[14]

2. H_0: $\mu = 100$, H_1: $\mu > 100$ — B1

$\bar{x} = \frac{710.9}{7} = 101.5571\ldots$; $s^2 = \dfrac{72219.45 - \frac{(710.9)^2}{7}}{6}$ — B1, M1

$s^2 = 3.746\ldots$ or $s = 1.9355$ — A1

	test statistic $t = \dfrac{101.557 - 100}{\frac{1.936}{\sqrt{7}}}$ = awrt 2.13	M1 A1	
	t_6 5% 1-tail critical value = 1.943	B1 ft	
	Significant result. Reject H_0, there is evidence that the mean weight is more than 100g.	A1	8

[8]

3.	D = dry—wet H$_0$: $\mu_D = 0$, H$_1$: $\mu_D \neq 0$	B1	
	d : 0.6, –1, –1.9, –1.4, –1.3, 0.5, –1.6, –0.6, –1.8	M1	
	$\bar{d}$: $-\dfrac{8.5}{9} = -0.9\dot{4}$, $s_d^2 = \dfrac{15.03 - 9 \times (\bar{d})^2}{8}$ = 0.87527…	A1, A1	
	$t = \dfrac{-0.9\dot{4}}{\frac{s_d}{\sqrt{9}}}$ = awrt –3.03	M1, A1	
	t_8 2-tail 1% critical value = 3.355	B1	
	Not significant – insufficient evidence of a difference between mean strength	A1 ft	8

[8]

4.	(a)		H$_0$: p = 0.3 (or 0.7) H$_1$: p < 0.3 (or > 0.7)	B1	1
	(b)		Let X = number who contract virus. Under H$_0$ $X \sim$ B(30, 0.3)		
			P(Type I error) = P($X < 6 \mid p = 0.30$) = P($X \leq 5$) = 0.0766	M1 A1	2
	(c)	(i)	Power = P($Y \leq 5 \mid Y \sim$ B(30, 0.2)) = 0.4275	M1 A1	
		(ii)	Power = P($Y \leq 5 \mid Y \sim$ B(30, 0.1)) = 0.9268	A1	3

(d)	Let C = number who contract virus. Under H_0 $C \sim B(50, 0.3)$			
	We require c such that $P(C \le c) \approx 0.05$	M1		
	$P(C \le 10) = 0.0789$, $P(C \le 9) = 0.0402$ $\therefore$ critical region is $C \le 9$	A1	2	
(e)	Size = 0.0402	B1	1	
(f)	(i) Power = $P(D \le 9 \mid D \sim B(50, 0.2))$ = 0.4437	B1		
	(ii) Power = $P(D \le 9 \mid D \sim B(50, 0.1))$ = 0.9755	B1	2	
(g)	Advantage: second test is more powerful			
	Disadvantage: second test involves greater sample size,	B1		
	$\therefore$ more expensive or takes longer	B1	2	
				[13]

5. (a) $\bar{x} = \dfrac{847.89}{10} = 84.79$; $s_x^2 = \dfrac{103712.6151 - (847.89)^2/10}{9}$ B1

$s_x^2 = 3535.6522...$ B1

or $s_x = 59.461\ldots$

2.262 B1

95% confidence interval for $\mu = 84.79 \pm 2.262 \times \dfrac{59.461}{\sqrt{10}}$ = (42.25, 127.33)

accept (42.3, 127.3) M1, A1, A16

(b) 95% confidence interval $\bar{x} \pm 1.96 \dfrac{\sigma}{\sqrt{n}}$

so chief accountant requires $1.96 \dfrac{\sigma}{\sqrt{n}} < 10$ M1 A1

i.e. $\dfrac{\sigma^2}{n} < \left(\dfrac{10}{1.96}\right)^2 = 26.0308\ldots = 26.03$ (2 d.p.) A1 cso 3

(c) Require the upper confidence limit of 98% confidence interval for σ^2

$\chi_9^2 = 2.088$; i.e. $\dfrac{9s^2}{\sigma^2} > 2.088$, $\Rightarrow \sigma^2 < 15239.88\ldots$

awrt 15240 B1; M1, A13

(d) Substitute into part (*b*), $n > \dfrac{15240}{26.03} \Rightarrow n = 586$ M1, A1 2

[14]

6. (a) (i) H_0: $\sigma_C^2 = \sigma_N^2$, H_1: $\sigma_C^2 > \sigma_N^2$ B1

$\dfrac{s_C^2}{s_N^2} = \dfrac{5.7^2}{3.5^2} = 2.652\ldots$; $F_{8,\,9}$ (5%) critical value = 3.23M1; B1

Not significant so do not reject H_0 – insufficient evidence that variance using conventional method is greater A1 ft 4

(ii) H_0: $\mu_N = \mu_C$ H_1: $\mu_N > \mu_C$ B1

$s^2 = \dfrac{8\times5.7^2 + 9\times3.5^2}{17} = \dfrac{370.17}{17} = 21.774\ldots$ M1

Test statistic $t = \dfrac{82.3 - 78.2}{\sqrt{21.774\ldots(\frac{1}{9}+\frac{1}{10})}} = 1.9122\ \ldots$

awrt 1.91 M1 A1

t_{17} (5%) 1-tail critical value = 1.740 B1

Significant – reject H_0. There is evidence that new style leads to an increase in mean A1 ft 6

(b) Assumed population of marks obtained were normally distributed B1 1

(c) Unbiased estimate of common variance is s^2 in (ii)

$7.564 < \dfrac{17s^2}{\sigma^2} < 30.191$ B1 M1 B1

$\sigma^2 > \dfrac{17\times21.774\ldots}{30.191} = 12.3$ (1 d.p.) A1

$\sigma^2 < \dfrac{17\times21.774\ldots}{7.564} = 48.9$ (1 d.p.) A1 5

[16]

7.	(a)	$X_1 \sim \mathrm{B}(10, p)$ $\therefore$ $\mathrm{E}(X_1) = 10p \Rightarrow \mathrm{E}(R_1) = \mathrm{E}\left(\frac{X_1}{10}\right) = \frac{10p}{10} = p$	B1	1
	(b)	$X_2 \sim \mathrm{B}(n, p)$ $\therefore$ $\mathrm{E}(X_2) = np \Rightarrow \mathrm{E}(R_2) = \mathrm{E}\left(\frac{X_2}{n}\right) = \frac{np}{n} = p$	B1	
		$\mathrm{E}(Y) = \mathrm{E}(\frac{1}{2}[R_1 + R_2]) = \frac{1}{2}[\mathrm{E}(R_1) + \mathrm{E}(R_2)] = \frac{1}{2}[p + p] = p$	B1	2
	(c)	$\mathrm{Var}(R_2) = \frac{1}{n^2}\mathrm{Var}(X_2) = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$	B1	
		$\mathrm{Var}(R_1) = \frac{p(1-p)}{10}$ $\therefore$ $\mathrm{Var}(Y) = \frac{1}{4}[\mathrm{Var}(R_1) + \mathrm{Var}(R_2)]$,	M1	
		$= \frac{1}{4}\left[\frac{p(1-p)}{10} + \frac{p(1-p)}{n}\right]$	A1	3
	(d)	Since $\mathrm{Var}(R_2) = \frac{p(1-p)}{n} \to 0$ as $n \to \infty$, $\therefore$ R_2 is consistent	M1, A1	2
	(e)	$\mathrm{Var}(R_1) = \frac{p(1-p)}{10} > \frac{p(1-p)}{20} = \mathrm{Var}(R_2)$	M1	
		$\mathrm{Var}(Y) = \frac{p(1-p)}{4}\left[\frac{1}{10} + \frac{1}{20}\right] = \frac{p(1-p)}{80} \times 3 < \mathrm{Var}(R_2)$		
		Since all 3 are unbiased, we select the one with minimum variance, i.e. Y	A1	2
	(f)	$X_1 + X_2 \sim \mathrm{B}(n + 10, p)$ so consider $\frac{X_1 + X_2}{n + 10}$	B1	
		$\mathrm{E}\left(\frac{X_1 + X_2}{n+10}\right) = \frac{(n+10)p}{(n+10)} = p$		
		(show unbiased)	M1	
		$\mathrm{Var}\left(\frac{X_1 + X_2}{n+10}\right) = \frac{p(1-p)}{n+10}$		
		(find variance)	M1	
		$\frac{p(1-p)}{n+10} < \frac{p(1-p)}{10}$ $\therefore$ always better than R_1		

And both A1

$\frac{p(1-p)}{n+10} < \frac{p(1-p)}{n}$ $\quad \therefore$ always better than R_2

$\frac{p(1-p)}{n+10} < \frac{p(1-p)}{4}\left[\frac{n+10}{10n}\right]$

$\Leftrightarrow \quad 40n < 100 + 20n + n^2$

$\Leftrightarrow \quad 0 < 10^2 - 20n + n^2$

$\Leftrightarrow \quad 0 < (10 - n)^2$

Show better than Y
Use of n = *20 acceptable* M1

$\therefore \ \frac{X_1 + X_2}{n+10}$ is unbiased and always has smaller variance A1 cso 6

[16]

 S4 Discrete distributions - Poisson

1. This question proved to be the most challenging question for many candidates. Few candidates wrote down the distributions of X_1, X_2 and X_3 in part (a) and those who tried were unable to do so accurately. The Normal and Binomial distributions were commonly seen. This aside candidates were then able to progress and gain at least two marks in part (a).

In part (b) the main error was not to use their means from part (a). Even the candidates who correctly identified the Poison introduced a variety of variances including σ^2.

In part (c) and (d) the candidates who knew E($\bar{Y}$) = μ and Var ($\bar{Y}$) = $\frac{\sigma^2}{n}$ gained full marks.

In part (e) the majority of candidates used Var$\left(\frac{1}{4}\bar{Y}\right)$ < Var ($\hat{\lambda}$) although it was not always clear from their working that this was the case with many writing Var$\left(\frac{1}{4}\bar{Y}\right)$ = Var ($\hat{\lambda}$) and simply solving the equation.

2. No Report available for this question.

3. No Report available for this question.

4. No Report available for this question.

5. No Report available for this question.

6. No Report available for this question.

7. No Report available for this question.